Section 7.7:

Transforms of Periodic and Power Functions

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Periodic Functions

Periodic Function

Definition 7. A function $f(t)$ is said to be **periodic of period T** $(\neq 0)$ if

$$f(t+T) = f(t)$$

for all t in the domain of f.

Discussion

- What is a periodic function
- Sine/Tangent/general periodic function
- Book pics
- "Windowed" version of a periodic function

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Periodic Functions

$$f(t) := \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \end{cases} \quad \text{and } f(t) \text{ has period 2.}$$

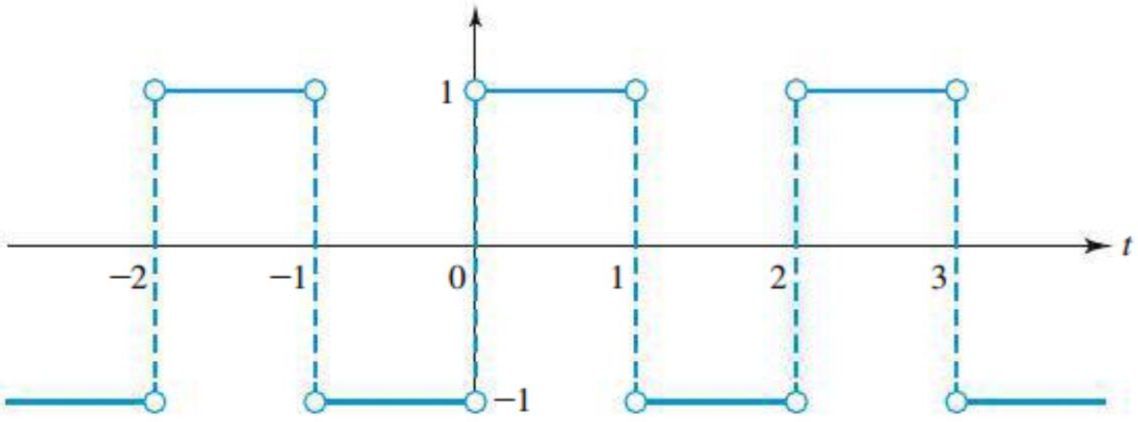

Figure 7.19 Graph of square wave function $f(t)$

Discussion

- What is a periodic function
- Sine/Tangent/general periodic function
- Book pics
- "Windowed" version of a periodic function

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Periodic Functions

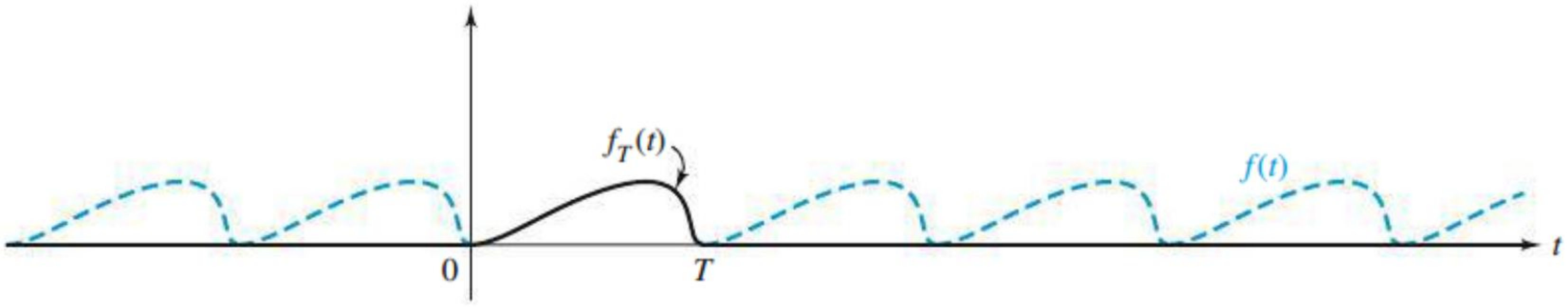

Figure 7.20 Windowed version of periodic function

Discussion

- What is a periodic function
- Sine/Tangent/general periodic function
- Book pics
- "Windowed" version of a periodic function

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Periodic Functions

It is convenient to introduce a notation for the "windowed" version of a periodic function $f(t)$, using a rectangular window whose width is the period:

$$f_T(t) := f(t)\Pi_{0,T}(t) = f(t)[u(t) - u(t-T)] = \begin{cases} f(t), & 0 < t < T, \\ 0, & \text{otherwise}. \end{cases} \qquad (2)$$

Discussion

- What is a periodic function
- Sine/Tangent/general periodic function
- Book pics
- "Windowed" version of a periodic function

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Periodic Functions

Transform of Periodic Function

Theorem 9. If f has period T and is piecewise continuous on $[0, T]$, then the Laplace transforms

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad \text{and} \quad F_T(s) = \int_0^\infty e^{-st} f_T(t)\, dt = \int_0^T e^{-st} f(t)\, dt$$

are related by

$$\text{(3)} \qquad F_T(s) = F(s)\left[1 - e^{-sT}\right] \quad \text{or} \quad F(s) = \frac{F_T(s)}{1 - e^{-sT}}.$$

Proof:

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Periodic Functions

Example 1 Determine $\mathscr{L}\{f\}$, where f is the periodic square wave function $f(t) := \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \end{cases}$ and $f(t)$ has period 2.

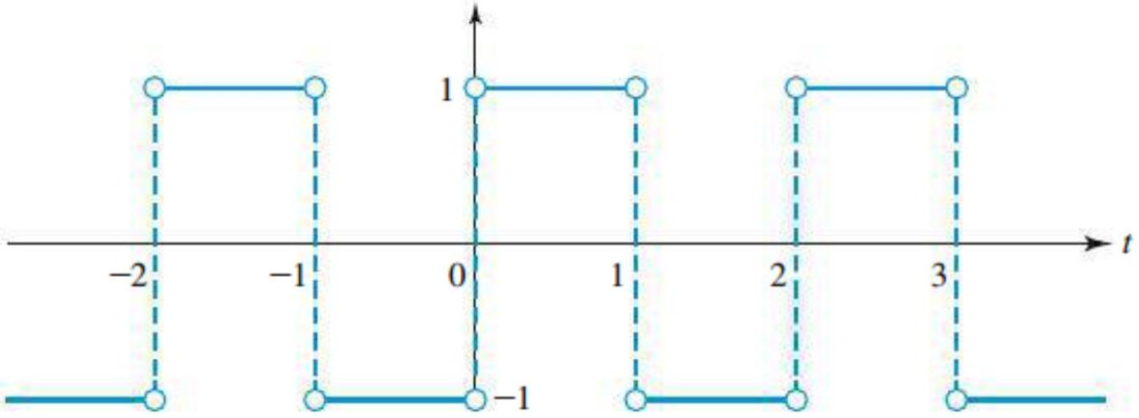

Figure 7.19 Graph of square wave function $f(t)$

$$F(s) = \frac{F_T(s)}{1 - e^{-sT}}.$$

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Series)

Discussion

- $\mathcal{L}\{\sum_{n=0}^{\infty} c_n t^n\} = \sum_{n=0}^{\infty} \mathcal{L}\{c_n t^n\}$

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Series)

Example 2 Determine $\mathscr{L}\{f\}$, where $f(t) := \begin{cases} \dfrac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Non-Integer Powers)

Discussion

- $\mathcal{L}\{\,t^n\,\} = \frac{n!}{s^{n+1}}$ doesn't make sense if n isn't an integer

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Non-Integer Powers)

Gamma Function

Definition 8. The **gamma function** $\Gamma(r)$ is defined by

(6) $$\Gamma(r) := \int_0^\infty e^{-u} u^{r-1} du, \qquad r > 0.$$

Notes

- This integral converges for all $r > 0$
- Recursive formula: $\Gamma(r+1) = r\Gamma(r)$
- $\Gamma(1) = 1$
- When n is an integer, $\Gamma(n+1) = n!$
- $\mathcal{L}\{ t^r \} = \frac{\Gamma(r+1)}{s^{r+1}}$ is true for all $r > -1$ even when r is NOT an integer

Discussion

- $\mathcal{L}\{ t^n \} = \frac{n!}{s^{n+1}}$ doesn't make sense if n isn't an integer

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Non-Integer Powers)

Gamma Function

Definition 8. The **gamma function** $\Gamma(r)$ is defined by

$$\text{(6)} \qquad \Gamma(r) := \int_0^\infty e^{-u}u^{r-1}\,du\,, \qquad r > 0\,.$$

Proofs: $\Gamma(r+1) = r\Gamma(r)$

Discussion

- $\mathcal{L}\{\, t^n \,\} = \frac{n!}{s^{n+1}}$ doesn't make sense if n isn't an integer

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Non-Integer Powers)

Gamma Function

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(6) $$\Gamma(r) := \int_0^\infty e^{-u} u^{r-1}\, du\,, \qquad r > 0\,.$$

Proofs: $\Gamma(1) = 1$

Discussion

- $\mathcal{L}\{\, t^n \,\} = \frac{n!}{s^{n+1}}$ doesn't make sense if n isn't an integer

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Non-Integer Powers)

Gamma Function

Definition 8. The **gamma function** $\Gamma(r)$ is defined by

$$\text{(6)} \qquad \Gamma(r) := \int_0^\infty e^{-u} u^{r-1} du\,, \qquad r > 0\,.$$

Proofs: When n is an integer, $\Gamma(n+1) = n!$

Discussion

- $\mathcal{L}\{\, t^n \,\} = \frac{n!}{s^{n+1}}$ doesn't make sense if n isn't an integer

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Non-Integer Powers)

Gamma Function

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(6) $$\Gamma(r) := \int_0^\infty e^{-u} u^{r-1} du, \qquad r > 0.$$

Discussion

- $\mathcal{L}\{ t^n \} = \frac{n!}{s^{n+1}}$ doesn't make sense if n isn't an integer

Proofs: $\mathcal{L}\{ t^r \} = \frac{\Gamma(r+1)}{s^{r+1}}$ is true for all $r > -1$ even when r is NOT an integer

Section 7.7: Transforms of Periodic and Power Functions

Transforms of Power Functions (Non-Integer Powers)

Example 3 Given that $\Gamma(1/2) = \sqrt{\pi}$, find the Laplace transform of $f(t) = t^{3/2}e^{2t}$.